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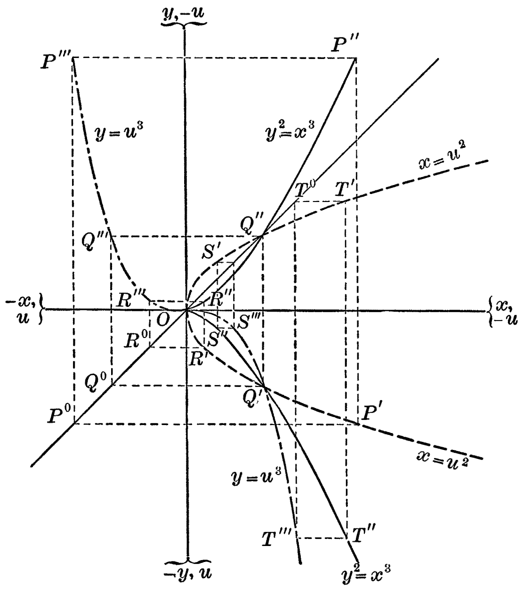

Fig. 5.

constructed. However, the construction under consideration often enables one to determine the salient features of the graph of $y = \phi[\psi(x)]$ from a mere rough plotting of the graphs of $y = \phi(u)$ and $u = \psi(x)$.[1]

A NOTE ON THE SUM OF THE REMAINDERS OF A SERIES.

By GLENN JAMES, Purdue University.

Suppose the series

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots \tag{1}$$

has the sum S, and consider the series formed from its remainders, which is

$$S + [S - a_0] + [S - (a_0 + a_1)] + \cdots + [S - (a_0 + a_1 + \cdots a_{n-1})] + \cdots \tag{2}$$

or

$$R_0 + R_1 + R_2 + \cdots + R_n + \cdots. \tag{3}$$

The assumption that (1) converges is equivalent to the assumption that

$$\lim_{n=\infty} R_n = 0.$$

Moreover, any convergent or divergent series

$$H_0 + H_1 + H_2 + \cdots + H_n + \cdots,$$

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[1] It might be remarked that the addition of the 45° line b on ordinary rectangular plotting paper would make of that paper a splendid medium for carrying out the above construction.

for which $\lim\limits_{n=\infty} H_n = 0$, is the remainder series of

$$(H_0 - H_1) + (H_1 - H_2) + \cdots + (H_{n-1} - H_n) + \cdots,$$

which converges since its sum is H_0.

Hence, there is an infinite class of convergent series whose remainder series converge and an infinite class whose remainder series diverge. The following theorems determine, upon the basis of the simpler convergence tests, certain types of series which belong to the former class.

THEOREM I. *Any series whose terms alternate in sign, never increase in absolute value, and tend to zero, possesses an absolutely convergent remainder series if the original series converges absolutely.*

Given an absolutely convergent series,

$$a_0 - a_1 + a_2 - a_3 + \cdots, \tag{4}$$

where

$$a_0 \geqq a_1 \geqq a_2 \geqq a_3 \geqq \cdots \geqq 0 \qquad \text{and} \qquad \lim_{n=\infty} a_n = 0.$$

The sum of such a series lies between zero and the first term, and the nature of such a series is not altered by dropping any number of consecutive terms from the beginning of the series. Also $S - S_{n-1}$, $(= R_n)$, is the sum of (4) after the $(n-1)$th term. Hence,

$$|R_n| < |a_n|, \qquad n = 0, 1, 2, 3, \cdots.$$

By hypothesis $\sum\limits_0^\infty a_n$ is absolutely convergent. Therefore, $\sum\limits_0^\infty R_n$ is absolutely convergent.

The following example shows that a series whose terms alternate in sign, continually decrease numerically, and tend to zero, does not necessarily have a convergent remainder series:

$$S = (\tfrac{1}{3} + \tfrac{3}{4}) - (\tfrac{1}{3} + \tfrac{3}{8}) + (\tfrac{1}{4} + \tfrac{3}{16}) - (\tfrac{1}{4} + \tfrac{3}{32}) + (\tfrac{1}{5} + \tfrac{3}{64}) - (\tfrac{1}{5} + \tfrac{3}{128}) \cdots = \tfrac{1}{2}. \tag{5}$$

This series does not converge absolutely, for each term is greater in absolute value than the corresponding term of the divergent series $\sum\limits_3^\infty (1/n)$. Noting that $R_1 = R_0 - a_0$, $R_2 = R_1 - a_1$, $\cdots$, the remainder series is evidently

$$\tfrac{1}{2} - (\tfrac{1}{3} + \tfrac{1}{4}) + \tfrac{1}{8} - (\tfrac{1}{4} + \tfrac{1}{16}) + \tfrac{1}{32} - (\tfrac{1}{5} + \tfrac{1}{64}) + \tfrac{1}{128} - \cdots$$

or

$$\tfrac{1}{2} - (\tfrac{1}{3} + \tfrac{1}{8}) - (\tfrac{1}{4} + \tfrac{1}{32}) - (\tfrac{1}{5} + \tfrac{1}{128}) - \cdots,$$

which plainly diverges.

THEOREM II. *If a series converges by the ratio test, its remainder series converges.*

Consider first a series $\sum\limits_0^\infty a_n$ for which $\lim\limits_{n=\infty} |a_n/a_{n-1}| = \alpha < 1$. Then for any positive ϵ there exists an n' such that

$$\alpha + \epsilon > \left|\frac{a_n}{a_{n-1}}\right| > \alpha - \epsilon, \qquad n \geqq n'.$$

Hence to every $n' + i$ there corresponds an ϵ_i, $|\epsilon_i| < 2\epsilon$, $(i = 1, 2, 3, \cdots)$, such that

$$\left|\frac{a_{n'}}{a_{n'-1}}\right| = \left|\frac{a_{n'+1}}{a_{n'}}\right| + \epsilon_1 = \left|\frac{a_{n'+2}}{a_{n'+1}}\right| + \epsilon_2 = \cdots.$$

Let the ratio $|a_{n'}/a_{n'-1}|$ be denoted by $K_{n'}$, and collect terms. Then

$$(6) \qquad K_{n'} = \left|\frac{a_{n'}}{a_{n'-1}}\right| = \frac{|a_{n'+1}| + \epsilon_1 |a_{n'}|}{|a_{n'}|} = \frac{|a_{n'+2}| + \epsilon_2 |a_{n'+1}|}{|a_{n'+1}|} = \cdots.$$

Whence

$$\begin{aligned} K_{n'}|a_{n'-1}| &= |a_{n'}|, \\ K_{n'}|a_{n'}| &= |a_{n'+1}| + \epsilon_1 |a_{n'}|, \\ K_{n'}|a_{n'+1}| &= |a_{n'+2}| + \epsilon_2 |a_{n'+1}|, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Adding these identities, we have

$$K_{n'}[|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots] = [|a_{n'}| + |a_{n'+1}| + |a_{n'+2}| + \cdots] \\ + [\epsilon_1 |a_{n'}| + \epsilon_2 |a_{n'+1}| + \cdots].$$

Whence, making use of (6),

$$(7) \quad K_{n'} = \frac{|a_{n'}| + |a_{n'+1}| + |a_{n'+2}| + \cdots}{|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots} + \frac{\epsilon_1 |a_{n'}| + \epsilon_2 |a_{n'+1}| + \cdots}{|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots} = \left|\frac{a_{n'}}{a_{n'-1}}\right|.$$

Now

$$|[\epsilon_1 |a_{n'}| + \epsilon_2 |a_{n'+1}| + \cdots|] < 2\epsilon[|a_{n'}| + |a_{n'+1}| + \cdots] < 2\epsilon[|a_{n'-1}| + |a_{n'}| + \cdots].$$

Hence

$$\left|\frac{\epsilon_1 |a_{n'}| + \epsilon_2 |a_{n'+1}| + \cdots}{|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots}\right| < 2\epsilon.$$

But $\lim\limits_{n'=\infty} 2\epsilon = 0$, which gives

$$(8) \qquad \mathop{L}_{n'=\infty} K_{n'} = \mathop{L}_{n'=\infty} \frac{|a_{n'}| + |a_{n'+1}| + |a_{n'+2}| + \cdots}{|a_{n'-1}| + |a_{n'}| + |a_{n'+1}| + \cdots} = \mathop{L}_{n'=\infty} \left|\frac{a_{n'}}{a_{n'-1}}\right| = \alpha.$$

Let $R_n' = |a_n| + |a_{n+1}| + \cdots$, and $R_n = a_n + a_{n+1} + \cdots$. Then $\sum\limits_0^\infty R_n'$ converges by (8). But $|R_n| \leqq R_n'$. Therefore $\sum\limits_0^\infty R_n$ converges.

Consider now a series $\sum\limits_0^\infty a_n$ for which the ratio a_n/a_{n-1} approaches no limit but remains less than α which is less than unity for all values of n greater than n'.

Then the series $\sum\limits_0^\infty a_n$ and its remainder series $\sum\limits_0^\infty R_n$ are dominated respectively by the series

$$(9) \qquad \sum_0^\infty A_n = \sum_0^{n'-1} |a_n| + \sum_{m=0}^\infty |a_{n'}| \alpha^m$$

and its remainder series $\sum_{0}^{\infty} \bar{R}_n$. But (9) is a series to which the above proof applies. Hence $\sum_{0}^{\infty} \bar{R}_n$ and consequently $\sum_{0}^{\infty} R_n$ converge.

Corollary. *If the ratio of the nth to the* $(n - 1)$*th term of a convergent series of positive terms has a limit, the corresponding ratio for the remainder series has the same limit.*

If $\sum_{0}^{\infty} a_n$ converges, the first part of the proof of the above theorem does not require that α, $(= \lim_{n=\infty} |a_n/a_{n-1}|)$, be less than unity. The corollary follows, then, from (8).

Examples (*a*), (*b*) and (*c*), below, illustrate the preceding theorem and corollary.

(*a*)

$$S = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 2.$$

The remainder series is

$$S' = 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 3.$$

(*b*) The geometric series

(i)

$$\sum_{0}^{\infty} ar^n = \frac{a}{1 - r}, \qquad |r| < 1,$$

has the remainder series

(ii)

$$\sum_{n=0}^{\infty} [ar^n + ar^{n+1} + \cdots] = \sum_{0}^{\infty} \frac{ar^n}{1 - r} = \frac{a}{(1 - r)^2}.$$

The remainder series of (ii) is

(iii)

$$\sum_{0}^{\infty} \frac{ar^n}{(1 - r)^2} = \frac{a}{(1 - r)^3}.$$

Thus, the *m*th remainder series of (i) is

$$\sum_{n=0}^{\infty} \frac{ar^n}{(1 - r)^m} = \frac{a}{(1 - r)^{m+1}}, \qquad |r| < 1.$$

Hence the sums of the successive remainder series increase indefinitely when $1 > r > 0$ and approach zero when $0 > r > -1$.

(*c*)

$$S = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)} + \cdots.$$

The remainder series is

$$S' = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots,$$

whence

$$\lim_{n=\infty} \frac{a_n}{a_{n-1}} = L \frac{n - 1}{n + 1} = L \frac{R_n}{R_{n-1}} = L \frac{n - 1}{n} = 1.$$

A further classification of series, as to the convergence or divergence of their remainder series, might be made upon the basis of other convergence tests. And in the case of series with variable terms it would be of interest to study such properties as uniform convergence with reference to their remainder series.